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Some Upper Bounds for 2-domination Number of Graphs and a Graffiti.pc Conjecture

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Abstract: Let $G = (V(G), E(G))$ be a graph and $k \geq 1$ integer. A subset $S \subseteq V(G)$ is called a k-dominating set of G if each vertex of $V(G) \setminus S$ is adjacent to at least k vertices of S in G. The k-domination number of G, denoted by $\gamma_k(G)$, is the cardinality of a minimum k-dominating set of G. Let $d_1 \leq d_2 \leq \cdots \leq d_n$ be the degree sequence of G. The upper median $m(G)$ of the degree sequence is defined as $m(G) = d_{\frac{n+1}{2}}$ if n is odd and $m(G) = d_{\frac{n}{2}+1}$ if n is even. An attractive conjecture of Graffiti.pc still remains open: for any connected graph G of order n, $\gamma_2(G) \leq n - m(G) + 1$. First we show a weaker version of the conjecture: $\gamma_2(G) \leq n - \delta(G) + 1$. By extending a known result on bipartite graphs, we show that for any triangle-free graph G with $\delta(G) \geq 2$, $\gamma_2(G) \leq n - \Delta(G)$, where $\Delta(G)$ is the maximum degree of G. It is a folklore that any graph G of size m and order n with $m \geq n$ contains a cycle. By extending it to hypergraphs, we are able to verify the validity of the Graffiti.pc conjecture for all split graphs.

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0 Introduction

Let $G = (V, E)$ be a finite, undirected, simple graph. The order and the size of G are $|V|$ and $|E|$, respectively. For a vertex $v \in V$, the neighborhood and the closed neighborhood of v are denoted by $N_G(v)$ and $N_G[v]$, respectively. The degree of a vertex v is denoted by $d_G(v)$. If there is no confusion, $N_G(v), N_G[v], d_G(v)$ are denoted by $N(v), N[v], d(v)$, repectively. The maximum degree and the minimum degree of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively.

Let $k \geq 1$ be an integer. A set $S \subseteq V$ is called a *k-dominating* set of G if each vertex of G is either in S or adjacent to at least k vertices in S. The k-domination number of G, denoted by $\gamma_k(G)$, is the cardinality of a minimum k-dominating set of G. A subset $S \subseteq V$ is called a k-independent set of G if $\Delta(G[S]) < k$. The k-independence number of G, denoted by $\alpha_k(G)$, is the cardinality of a maximum k-independent set of G. Observe that $\gamma_1(G)$ and $\alpha_1(G)$ are precisely the domination number $\gamma(G)$ and the independence number $\alpha(G)$, respectively. Favoron[7] proved that for any graph G and any integer $k \geq 1$, $\gamma_k(G) \leq \alpha_k(G)$. DeLaViña et

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al.[5] showed that $\gamma_k(G) \leq \frac{r}{r+1}n$ for any graph G of order n with $\delta(G) \geq \frac{r+1}{r}k - 1$. We refer to [3] for more results and open problems on k-domination of graphs.

We focus on 2-domination number of graphs in the paper. Some known results on it are listed as follows.

- If $\delta(G) \geq 2$, then $\gamma_2(G) \leq \frac{2}{3}n$ (see [8]).
- If $\delta(G) \geq 3$, then $\gamma_2(G) \leq \frac{1}{2}n$ (see [2]).
- If $\delta(G) \geq 6$, then $\gamma_2(G) < \frac{1}{2}n$ (see [1]).
- For any graph G with minimum degree $\delta \geq 0$, $\gamma_2(G) \leq \frac{2\ln(\delta+1)+1}{\delta+1}n$ (see [10]).
- For any graph G, $\gamma_2(G) + \gamma_2(\overline{G}) \leq n + 2$ (see [11]).
- For any bipartite graph G, $\gamma_2(G) \leq \frac{3}{2}\alpha(G)$ (see [9]).

An attractive conjecture, which is known as Graffiti.pc Conjecture 387 in [4], still remains to be unsolved. Main results of the current paper are devoted to the conjecture. Let $d_1 \leq d_2 \leq \cdots \leq d_n$ be the degree sequence of G. The upper median $m(G)$ of the degree sequence is defined as $m(G) = d_{\frac{n+1}{2}}$ if n is odd and $m(G) = d_{\frac{n}{2}+1}$ if n is even.

Conjecture 0.1[4] (Graffiti.pc Conjecture 387) For any connected graph of order n, $\gamma_2(G) \leq n - m(G) + 1$.

First we show a weaker version of the conjecture: $\gamma_2(G) \leq n - \delta(G) + 1$. By extending a result of DeLaViña et al. [6] on bipartite graphs, we show that for any triangle-free graph G with $\delta(G) \geq 2$, $\gamma_2(G) \leq n - \Delta(G)$.

It is well known that any graph G of size m and order n with $m \geq n$ contains a cycle. By extending it to hypergraphs, we verify the validity of the Graffiti.pc conjecture for all split graphs.

1 Some New Bounds

By the definition, since $\delta(G) \leq m(G) \leq \Delta(G)$ for any graph G,

$$n - \Delta(G) + 1 \leq n - m(G) + 1 \leq n - \delta(G) + 1.$$

First we begin with a weaker result.

Theorem 1.1 For any graph G of order n, $\gamma_2(G) \leq n - \delta(G) + 1$.

Proof The result trivially holds when $\delta(G) \leq 1$. So, assume that $\delta(G) \geq 2$. Take a vertex u with $d(u) = \delta(G) = \delta$. Let $S = V(G) \setminus N[u]$.

If for every vertex $v \in N(u)$, $N(v) \cap S \neq \emptyset$, one can see that $S \cup \{u\}$ is a 2-dominating set of G. It follows that

$$\gamma_2(G) \leq |S \cup \{u\}| = n - \delta.$$

On the other hand, if there exists a vertex $v \in N(u)$ with $N(v) \cap S = \emptyset$, then $d(v) = \delta$. Equivalently, $N[v] = N[u]$. Hence, $S \cup \{u, v\}$ is a 2-dominating set of G, and thus

$$\gamma_2(G) \leq |S \cup \{u, v\}| = n - \delta + 1.$$

□

For regular graphs, we have a stronger assertion than that of Conjecture 0.1.

Corollary 1.1 For a k-regular graph G of order n, $\gamma_2(G) \leq n - k + 1$, with equality if and only if $G \cong K_n$.

Proof If $G = K_n$, then $\gamma_2(G) = 2$ and $k = n - 1$. Clearly, $\gamma_2(G) = n - k + 1$.

Next assume that $G \neq K_n$. Fix a vertex $u \in V(G)$ and let $S = V(G) \setminus N[u]$. Set $M = \{v \in N(u) : N_G(v) \cap S = \emptyset\}$. If $M = \emptyset$, then $S \cup \{u\}$ is a 2-dominating set of G, and we are done, because

$$\gamma_2(G) \leq |S \cup \{u\}| = n - k.$$

Suppose $M \neq \emptyset$. Since G is not a complete graph, $N(u) \setminus M \neq \emptyset$. Take a vertex $x \in S$ such that x is adjacent to some vertices of $N(u)$.

Case 1: $|N_G(x) \cap S| \geq 2$.

One can see that $S \setminus \{x\} \cup \{u, v\}$ $(v \in M)$ is a 2-dominating set of G,

$$\gamma_2(G) \leq |S \setminus \{x\} \cup \{u, v\}| = n - k.$$

Case 2: $|N_G(x) \cap S| = 1$.

Let y be the unique element in $N_G(x) \cap S$ and let $v_0 (\in M)$ be the unique vertex of $N(u)$ which is not adjacent to x in G (see Fig. 1).

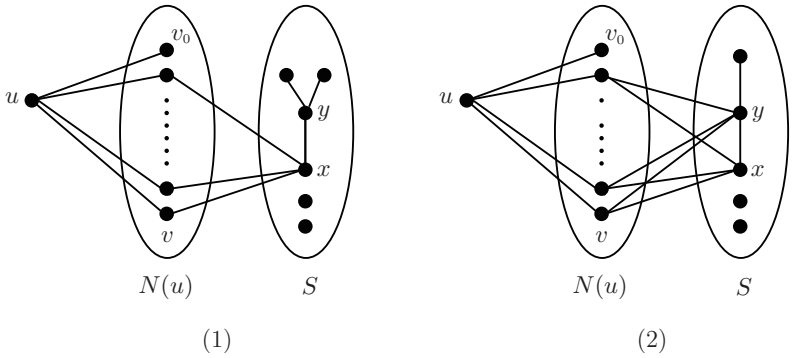


Figure 1 $|N(u)| = d(u) = m(G) = k$

Subcase 2.1: $|N_G(y) \cap S| \geq 2$.

Since $S \setminus \{y\} \cup \{u, v_0\}$ is a 2-dominating set of G,

$$\gamma_2(G) \leq |S \setminus \{y\} \cup \{u, v\}| = n - k.$$

Subcase 2.2: $|N_G(y) \cap S| = 1$.

Since $S \setminus \{y\} \cup \{v_0, v\}$ $(v \in N(u) \setminus M)$ is a 2-dominating set,

$$\gamma_2(G) \leq |S \setminus \{y\} \cup \{v_0, v\}| = n - k.$$

In summary, for any k-regular non-complete graph G, $\gamma_2(G) \leq n - k$. □

The following result is immediate from Theorem 1.1.

Corollary 1.2 For any graph G of order n, if $m(G) = \delta(G)$, then $\gamma_2(G) \leq n - m(G) + 1$.

Next we can move forward one more step.

Theorem 1.2 For a connected graph G of order n, if $m(G) \leq \delta(G) + 1$, $\gamma_2(G) \leq n - m(G) + 1$.

Proof In view of Corollary 1.2, assume that $m(G) = \delta(G) + 1$. Consider a vertex u with $d(u) = m(G)$. Let $S = V \setminus N[u]$. Set $M = \{v \in N(u) : N_G(v) \cap S = \emptyset\}$. Clearly, for each $v \in M$, $\delta(G) \leq d(v) \leq \delta(G) + 1$.

Case 1: $M = \emptyset$.

Clearly, $S \cup \{u\}$ is a 2-dominating set of G, and thus $\gamma_2(G) \leq n - m(G)$.

Case 2: $M \neq \emptyset$ and there exists a vertex $v_0 \in M$ with $d(v_0) = m(G)$.

Since $N[v_0] = N[u]$, $S \cup \{u, v_0\}$ is a 2-dominating set of G, and thus $\gamma_2(G) \leq n - m(G) + 1$.

Case 3: $M \neq \emptyset$, and $d(v) = \delta(G) = m(G) - 1$ for any $v \in M$.

Subcase 3.1: $|M| = 1$.

Let $M = \{v_0\}$. Clearly, $S \cup \{u, v_0\}$ is a 2-dominating set of G, and hence $\gamma_2(G) \leq |S \cup \{u, v_0\}| \leq n - m(G) + 1$.

Subcase 3.2: $|M| = 2$.

Let $M = \{v_1, v_2\}$. If $v_1v_2 \in E(G)$, either $S \cup \{u, v_1\}$ or $S \cup \{u, v_2\}$ is a 2-dominating set of G, and thus $\gamma_2(G) \leq |S \cup \{u, v_1\}| \leq n - m(G) + 1$.

Now assume that $v_1v_2 \notin E(G)$. If $N(u) \setminus M \neq \emptyset$, by choosing a vertex $v \in N(u) \setminus M$, we find a 2-dominating set $S \cup \{u, v\}$ of G. Again we have $\gamma_2(G) \leq |S \cup \{u, v\}| \leq n - m(G) + 1$. If $N(u) \setminus M = \emptyset$, and since G is connected, it follows that $G \cong K_{1,2}$. Clearly, $m(G) = 1 = \delta(G)$. The result is true.

Subcase 3.3: $|M| = t \geq 3$.

Let $M = \{v_1, \cdots, v_t\}$. If there exist two vertices $v_i, v_j \in M$ such that $v_iv_j \notin E(G)$, then $S \cup \{v_i, v_j\}$ is a 2-dominating set of G. So, $\gamma_2(G) \leq |S \cup \{v_i, v_j\}| \leq n - m(G) + 1$. Otherwise, M is a clique of G. In this case, $S \cup \{u, v_1\}$ is a 2-dominating set of G. Again, $\gamma_2(G) \leq |S \cup \{u, v_1\}| \leq n - m(G) + 1$.

In a word,

$$\gamma_2(G) \leq n - m(G) + 1.$$

□

In [6], the authors showed that $\gamma_2(G) \leq n - m(G) + 1$ for any bipartite graph G. Next we extend it to any triangle-free graph G with $\delta(G) \geq 2$ in a much stronger version.

Theorem 1.3 Let G be a triangle-free graph of order n. If $\delta(G) \geq 2$, then $\gamma_2(G) \leq n - \Delta(G)$.

Proof Consider a vertex u with $d(u) = \Delta(G)$. Let $S = V(G) \setminus N[u]$. Since G is a triangle-free graph, $N(u)$ is an independent set of G. Since $\delta = \delta(G) \geq 2$, for every vertex $v \in N(u)$, $N_G(v) \cap S \neq \emptyset$. Clearly, $S \cup \{u\}$ is a 2-dominating set of G, implying $\gamma_2(G) \leq n - \Delta(G)$. □

The bound of the above theorem is sharp. For instance, $\gamma_2(K_{2,n-2}) = 2 = n - (n-2) = n - \Delta(K_{2,n-2})$. However, the condition $\delta \geq 2$ is necessary, since $\gamma_2(K_{1,n-1}) = n - 1 > 1 = n - \Delta(K_{1,n-1})$.

2 Split Graphs

To verify Conjecture 0.1 for split graphs, we establish a result on hypergraphs, which is a natural extension of a folklore on graphs: any graph of size at least its order contains a cycle.

2.1 A Result on Hypergraphs

Let $H = (V, \mathcal{F})$ be a hypergraph. For an edge $F \in \mathcal{F}$, the edge-deleted subgraph, denoted by $H \setminus F$, is the hypergraph $(V(H), \mathcal{F} \setminus \{F\})$. For a family $\mathcal{F}'$ of subsets of V, $H + \mathcal{F}'$ denotes the hypergraph $(V, \mathcal{F} \cup \mathcal{F}')$. For a vertex $v \in V(H)$, there are two types of vertex-deleted hypergraph with vertex set $V(H) \setminus \{v\}$: $H - v$ and $H \ominus v$, the edge set of $H - v$ is $\mathcal{F} \setminus \{F \in \mathcal{F} : \ v \in F\}$, while the edge set of $H \ominus v$ is $\{F \setminus \{v\} : F \in \mathcal{F}\}$. Clearly, the $H \setminus F$ and $H - v$ is precisely the edge-deleted subgraph and the vertex deleted subgraph when H is a graph.

The rank and the anti-rank of H are $\min\{|F| : \ F \in \mathcal{F}\}$ and $\max\{|F| : \ F \in \mathcal{F}\}$, respectively. For an integer $k \geq 0$, $D_{k^+}(H) = \{v \in V : \ d_H(v) \geq k\}$. For a subset $\mathcal{F}' \subseteq \mathcal{F}$, the partial (or edge-induced) hypergraph induced by $\mathcal{F}'$, denoted by $H[\mathcal{F}']$, is the hypergraph $(\bigcup_{F \in \mathcal{F}'} F, \mathcal{F}')$.

Let H be a hypergraph of order n with anti-rank $t \geq 2$. We say that H admits $\mathscr{P}_k$*-property* for an integer $k \in \{2, \cdots, n + 2 - t\}$ if there exists k-subset $\mathcal{F}'$ of $\mathcal{F}$ which 2-dominates at least $k + t - 2$ vertices of H, that is, $|D_{2^+}(H[F'])| \geq k + t - 2$. For convenience, we denote by $H \in \mathscr{P}_k$ if H admits $\mathscr{P}_k$-property.

Remark 2.1 If $H \setminus F \in \mathscr{P}_k$ for an edge F of H, then $H \in \mathscr{P}_k$.

Remark 2.2 If $H - v \in \mathscr{P}_k$ for a vertex v of H, then $H \in \mathscr{P}_k$.

Remark 2.3 If $H \ominus v \in \mathscr{P}_k$ for a vertex v of H, then $H \in \mathscr{P}_k$.

Lemma 2.1 Let $H = (V, \mathcal{F})$ be a hypergraph of order n and size m, with anti-rank at least $t \geq 2$. If $m \geq n$, then $H \in \mathscr{P}_k$ for some $k \in \{2, \cdots, n + 2 - t\}$.

Proof The proof is induction on m. Since the anti-rank of H is at least t, $n \geq t$, and thus $m \geq t$. If $m = t$, then $n = t$. It follows that $F = V$ for any $F \in \mathcal{F}$. Therefore, each pair of edges of H can 2-cover V (t vertices), implying $H \in \mathscr{P}_2$.

Next assume that $m > t$. If $m - 1 \geq n$, then by the induction hypothesis, $H \setminus F \in \mathscr{P}_k$ for some $k \in \{2, \cdots, n + 2 - t\}$. By Remark 2.1, $H \in \mathscr{P}_k$. So, we consider the case when $m = n$. If there exists a vertex $y \in V$ with $d(y) \leq 1$, by the induction hypothesis, $H - y \in \mathscr{P}_k$ for some $k \in \{2, \cdots, n + 2 - t\}$. By Remark 2.2, $H \in \mathscr{P}_k$. So, let $\delta(H) \geq 2$, and we divide into two cases.

Case 1: $\delta(H) \geq t$.

Clearly, $\mathcal{F}$ can t-dominate V. Since $\delta(H) \geq t$, every $m - t + 2$ $(= n - t + 2)$ members of $\mathcal{F}$ can 2-dominate all vertices of V, equivalently, $H \in \mathscr{P}_{n-t+2}$.

Case 2: $\delta(H) = s$ for some $s \in \{2, \cdots, t - 1\}$.

Take a vertex y with $d(y) = s$, and let $y \in F_i$ for $i \in \{1, \cdots, s\}$. If $H[\{F_1, \cdots, F_s\}] \in \mathscr{P}_k$ for some $k \in \{2, \cdots, s\}$, then by Remarks 2.1–2.2, $H \in \mathscr{P}_k$. In sequel, suppose that $H[\{F_1, \cdots, F_s\}] \notin \mathscr{P}_k$ for every $k \in \{2, \cdots, s\}$.

Claim: There is a vertex $y' \in \bigcup_{i=1}^{s} F_i$ which belongs to a unique edge in $\{F_i : \ 1 \leq i \leq s\}$.

Suppose that the claim is not true, that is, $\{F_i : \ 1 \leq i \leq s\}$ is a 2-cover of $\bigcup_{i=1}^{s} F_i$. However, since $H[\{F_1, \cdots, F_s\}] \notin \mathscr{P}_s$,

$$\left| \bigcup_{i=1}^{s} F_i \right| \leq s + t - 3.$$

Let $\mathcal{F}_l$ be any l-subset of $\{F_1, \cdots, F_s\}$. We shall show that $|\bigcup_{F \in \mathcal{F}_l} F| \geq t + l - 1$ for each $l \in \{2, \cdots, s\}$. By induction on l. For the case when $l = 2$, letting $\mathcal{F}_l = \{F_i, F_j\}$, without loss of

generality, we show that $|F_i \cup F_j| \geq t+1$. Since $H[\{F_1, \cdots, F_s\}] \notin \mathscr{P}_k$ for any $k \in \{2, \cdots, s\}$, $H[\{F_1, F_2\}] \notin \mathscr{P}_2$. If $|F_i \cup F_j| \leq t$, by $|F_i| \geq t$ and $|F_j| \geq t$, then $F_i = F_j$ and $|F_i| = t$. By the definition, $H[\{F_1, F_2\}] \in \mathscr{P}_2$, a contradiction. This proves $|F_i \cup F_j| \geq t+1$.

Now assume that $l > 2$ and $|\bigcup_{F \in \mathcal{F}_{l-1}} F| \geq t+l-2$ for any $(l-1)$-subset $\mathcal{F}_{l-1}$ of $\{F_1, \cdots, F_s\}$. Let us consider any l-subset $\mathcal{F}_l$ of $\{F_1, \cdots, F_s\}$. Suppose that $|\bigcup_{F \in \mathcal{F}_l} F| \leq t+l-2$. Indeed, it follows that $\mathcal{F}_l$ is 2-cover of $\bigcup_{F \in \mathcal{F}_l} F$, implying that $H[\mathcal{F}_l] \in \mathscr{P}_l$, and hence $H[\{F_1, \cdots, F_s\}] \in \mathscr{P}_k$, a contradiction. This proves $|\bigcup_{F \in \mathcal{F}_l} F| \geq t+l-1$.

In particular,

$$\left| \bigcup_{i=1}^{s-1} F_i \right| \geq s+t-2.$$

However, $|\bigcup_{i=1}^{s-1} F_i| \leq |\bigcup_{i=1}^{s} F_i| \leq s+t-3$, contradiction again.

By the above claim, without loss of generality, let $y' \in F_1$.

Subcase 2.1: $H' = (H \setminus F_1) \ominus y \in \mathscr{P}_k$.

By Remark 2.4, $H \setminus F_1 \in \mathscr{P}_k$ and furthermore, by Remark 2.2, $H \in \mathscr{P}_k$.

Subcase 2.2: $H' = (H \setminus F_1) \ominus y \notin \mathscr{P}_k$.

Indeed, $H' = H - y + \mathcal{F}'$, where $\mathcal{F}' = \{F_i \setminus \{y\} : \ 2 \leq i \leq s\}$. Let $H'' = H - y + \mathcal{F}''$, where $\mathcal{F}'' = \{\{y'\} \cup F_i \setminus \{y\} : \ 2 \leq i \leq s\}$. Since H'' has order $n-1$ and size $m-1$, by the induction hypothesis, $H'' \in \mathscr{P}_k$. By the definition, there exist k edges $F_1'', \cdots, F_k''$, of H'' such that $|D_{2^+}(H''[\{F_1'', \cdots, F_k''\}])| \geq k+t-2$. Combining this with $H' = (H \setminus F_1) \ominus y \notin \mathscr{P}_k$, we conclude that $y' \in D_{2^+}(H''[\{F_1'', \cdots, F_k''\}])$ and $\{F_1'', \cdots, F_k''\} \cap \mathcal{F}'' \neq \emptyset$.

Subcase 2.2.1: $|\{F_1'', \cdots, F_k''\} \cap \mathcal{F}''| = 1$.

Without loss of generality, let $F_2'' = \{y'\} \cup F_2 \setminus \{y\}$. Moreover, $F_i'' \in \mathcal{F}$ for each $i \neq 2$. Since $y \in F_1 \cap F_2$ and $y' \in F_1$, $D_{2^+}(H[\{F_1, F_2\} \cup \{F_1'', \cdots, F_k''\} \setminus \{F_2''\}]) = D_{2^+}(H''[F_1'', \cdots, F_k'']) \cup \{y\}$, implying $H \in \mathscr{P}_{k+1}$.

Subcase 2.2.2: $|\{F_1'', \cdots, F_k''\} \cap \mathcal{F}''| = r \geq 2$.

Without loss of generality, let $F_i'' = \{y'\} \cup F_i \setminus \{y\}$ for each $i \in \{2, \cdots, r+1\}$. Since $y \in F_2 \cap \cdots \cap F_{r+1}$, we have

$$\begin{aligned} &\{y\} \cup D_{2^+}(H''[\{F_1'', \cdots, F_k''\}]) \setminus \{y'\} \\ &\quad \subseteq D_{2^+}(H[\{F_2, \cdots, F_{r+1}\} \cup \{F_1'', \cdots, F_k''\} \setminus \{F_2'', \cdots, F_{r+1}''\}]), \end{aligned}$$

implying $H \in \mathscr{P}_k$. □

Remark 2.4 It is interesting that for a simple graph G, $G \in \mathscr{P}_k$ means that G contains a cycle of length k. Hence, Lemma 2.1 implies that any graph G of size at least its order contains a cycle.

2.2 Split Graphs and Threshold Graphs

Now we are ready to give one of our main results. A graph G is called a *split graph* if $V(G)$ can be partitioned into two sets X and Y such that X is a clique and Y is an independent set of G.

Theorem 2.1 For any split graph G of order n, $\gamma_2(G) \leq n - m(G) + 1$.

Proof Let X and Y be a partition of $V(G)$ such that X is a maximal clique of G and Y is an independent set of G. By the maximality of X, $d(x) \geq |X| - 1 \geq d(y)$ for any vertices $x \in X$ and $y \in Y$.

Fact 1: $Y \cup \{x_1, x_2\}$ is a 2-dominating set of G for any two vertices $x_1, x_2 \in X$.

Case 1: $|X| < |Y|$.

It follows that a vertex with degree $m(G)$ belongs to Y. Since $d(y) \leq |X|-1$ for each $y \in Y$, $m(G) \leq |X| - 1$, and hence

$$\gamma_2(G) \leq |Y \cup \{x_1, x_2\}| = |Y| + 2 = n - |X| + 2 \leq n - m(G) + 1.$$

Case 2: $|X| \geq |Y|$.

It follows that a vertex with degree $m(G)$ belongs to X. Let $X_1 = \{x \in X |\ d(x) \geq m(G)\}$. Observe that $|X_1| \geq \frac{n}{2} \geq |Y|$. Suppose $m(G) = |X| - 1 + t,\ 0 \leq t \leq |Y|$.

First, let $t = 0$. Since $Y \cup \{x_1, x_2\}$ is a 2-dominating set of G for any two vertices $x_1, x_2 \in X$, we have $\gamma_2(G) \leq |Y| + 2 = n - |X| + 2 = n - (|X| - 1) + 1 = n - m(G) + 1$.

Second, let $t = 1$. There are two possible cases: $|X_1| = |X|$ and $|X_1| < |X|$. If $|X_1| = |X|$, then $Y \cup \{x\}$ is a 2-dominating set of G, where $x \in X$. Hence $\gamma_2(G) \leq |Y| + 1 = n - |X| + 1 = n - m(G) + 1$.

If $|X_1| < |X|$, then $|Y| < \frac{n}{2} \leq |X_1|$. There exists a vertex, say y_1, has two neighbors, say x, x' in X_1. Clearly, $(Y \setminus \{y_1\}) \cup \{x, x'\}$ is 2-dominating set of G. Thus,

$$\gamma_2(G) \leq |(Y \setminus \{y_1\}) \cup \{x, x'\}| = n - |X| + 1 = n - m(G) + 1.$$

It remains to tackle the case when $t \geq 2$. To this end, let $H = (Y, \mathcal{F})$ be the hypergraph on the vertex set Y, in which $\mathcal{F}$ consists of $F = N(x) \cap Y$ for each $x \in X_1$. Since $|X_1| \geq |Y|$, by Lemma 2.1, $H \in \mathscr{P}_k$ for some $k \in \{2, \cdots, |Y| - t + 2\}$.

Without loss of generality, there exists a k-set $X' \subseteq X$, which 2-dominate a $(k + t - 2)$-set Y' of Y. Obviously, $X' \cup (Y \setminus Y')$ is a 2-dominating set of G. We conclude that

$$\gamma_2(G) \leq k + |Y| - (k + t - 2) = |Y| - t + 2 = n - m(G) + 1.$$

The proof is complete. □

Let G be a graph with vertex set $V(G) = \{r_1, r_2, \cdots, r_n\}$, where r_i is a real number, in which two vertices r_i and r_j are adjacent if and only if $r_i + r_j > 0$. Such a graph is called a *threshold* graph. It is clear that every threshold graph is a split graph. We determine the 2-domination number of a threshold graph. Let $d_1, d_2, \cdots, d_n$ be the degree sequence of G, with nondecreasing order. We denote d_{n-1} by $\Delta_2(G)$, whereas $d_n = \Delta(G)$.

Theorem 2.2 For any threshold graph G of order n,

$$\gamma_2(G) = \begin{cases} n, & \text{if } \Delta(G) \leq 1, \\ n - 1, & \text{if } \Delta(G) \geq 2 \text{ and } \Delta_2(G) = 1, \\ n - \Delta_2(G) + 1, & \text{if } \Delta_2(G) \geq 2. \end{cases}$$

Proof Let $V(G) = \{r_i : 1 \leq i \leq n\}$. Without loss of generality, let $r_i \leq r_j$ for any two integers i, j with $i \leq j$. Clearly, $d_1 \leq d_2 \leq \cdots \leq d_n$, where $d_i = d(r_i)$ for each i.

If $\Delta(G) \leq 1$, then $\gamma_2(G) = n$.

If $\Delta_2(G) \geq 2$, then $\{r_{n-1}, r_n\} \cup S_1$ is a minimum 2-dominating set of G, where $S_1 = \{v : d(v) \leq 1\}$. Moreover, $\Delta_2(G) = d(r_{n-1}) = n-1-|S_1|$, and thus $\gamma_2(G) = 2+|S_1| = n-\Delta_2(G)+1$.

If $\Delta(G) \geq 2$ and $\Delta_2(G) = 1$, then $V \setminus \{r_n\}$ is the minimum 2-dominating set of G, $\gamma_2(G) = n-1$. □

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图的 2- 控制数的上界和一个 Graffiti.pc 猜想

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摘要: 设 $G = (V(G), E(G))$ 是一个图, k 是一个正整数. 称一个顶点子集 S 为 G 的 k- 控制集, 若 $V(G) \setminus S$ 中的每个顶点在 S 中至少有 k 个邻点, 我们用 $r_k(G)$ 表示 k- 控制集的最小阶数. 令 $d_1 \leq d_2 \leq \cdots \leq d_n$ 为图 G 的度序列. 当 n 为偶数时, 度序列中位数 $m(G) = d_{\frac{n}{2}+1}$, 当 n 为奇数时, 度序列中位数 $m(G) = d_{\frac{n+1}{2}}$. 一个仍未解决的 Graffiti.pc 猜想说: 对任一 n 个顶点的连通图 G, $r_2(G) \leq n-m(G)+1$. 首先我们证明了此猜想的一个弱形式: $r_2(G) \leq n-d_1+1$. 此外, 通过拓展此猜想在二部图上的结果, 我们证明了对最小度不小于 2 的无三角形图 G, $r_2(G) \leq n-\Delta(G)$, 其中 $\Delta(G)$ 为图 G 的最大度. 众所周知, 每一个其边数不少于顶点数的图都包含一个圈. 我们将此结论推广到超图上. 进而得到上述猜想对所有分裂图都成立.

关键词: 2- 控制数; 无三角形图; 分裂图; 超图